

# On the Mutual Information between disjoint regions in AdS/CFT

Javier Molina-Vilaplana

Department of Systems Engineering, Technical University Cartagena, Dr Fleming S/N 30202 Cartagena, Spain

The holographic mutual information between two separated circular regions in a  $3+1$  dimensional gauge theory dual to  $\text{AdS}_5 \times S^5$  through the AdS/CFT correspondence, is computed in the limit in which the separation  $L$  between the regions is much larger than their sizes  $a$ . The calculation uses some previous results concerning the holographic computation of the long distance correlator of two distant Wilson loops. It is shown that for these regimes, the holographic mutual information follows a power law decaying behaviour whose order is given by a prefactor  $\sqrt{\lambda}$ , with  $\lambda$  the t'Hooft coupling of the boundary theory. This contradicts a conjectured sharp vanishing of this quantity in the large separation regime. The result is compared with a recent prediction on the mutual information between distant regions in  $3+1$  dimensional free conformal field theories.

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## INTRODUCTION

Entanglement entropy (EE) and other related information-theoretic quantities such as mutual information (MI) are by now regarded as valuable tools to study different phenomena in quantum field theories and many body systems [1]. These quantities provide a new kind of information that cannot be obtained from more standard observables such as expectation values. Namely, both EE and MI, are sensitive probes able to detect non-local signatures of the theory such as topological order which can not be detected by any local observable. Concretely, the mutual information  $I_{AB}$  between two arbitrary regions  $A$  and  $B$  has certain advantages over the entanglement entropy. First,  $I_{AB}$  can be viewed as an entropic correlator between  $A$  and  $B$  defined by,

$$I_{AB} = S_A + S_B - S_{A \cup B}, \quad (1)$$

where  $S_{A,B}$  is the entanglement entropy of the region  $A(B)$  and  $S_{A \cup B}$  is the EE of the two regions. By its definition,  $I_{AB}$  is finite and, contrarily to EE, is non UV-cutoff dependent. In addition, the strong subadditivity property of the EE states that when  $A$  and  $B$  are disconnected, then

$$S_A + S_B \geq S_{A \cup B}, \quad (2)$$

which immediately leads to realize that  $I_{AB} \geq 0$ . A standard approach to compute EE and MI makes use of the replica trick [2]. Unfortunately, these calculations are notoriously difficult to carry out, even in the case of free field theories.

In the context of the AdS/CFT [3], however, Ryu and Takayanagi (RT) have recently proposed a remarkably simple formula [4] to obtain the EE of an arbitrary region  $A$  of a  $d+1$  dimensional CFT which admits a classical gravity dual given by an asymptotically  $\text{AdS}_{d+2}$  spacetime. According to the RT formula, the EE is obtained in terms of the area of a certain minimal surface  $\gamma_A$  in the dual higher dimensional gravitational geometry; as a result, the entanglement entropy  $S_A$  in a CFT $_{d+1}$  is given by the celebrated area law relation

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}, \quad (3)$$

where  $d$  is the number of spacetime dimensions of the boundary CFT and  $\gamma_A$  is the  $d$ -dimensional static minimal surface in  $\text{AdS}_{d+2}$  such that  $\partial A = \partial \gamma_A$ . The  $G_N^{(d+2)}$  is the  $d+2$  dimensional Newton constant.

In this letter we consider the mutual information between two disconnected regions  $A$  and  $B$  in the ground state of an strongly coupled quantum field theory which admits a gravity dual through the AdS/CFT correspondence. Using (3) in (1), this quantity reads,

$$I_{AB} = \frac{1}{4G_N^{(d+2)}} [\text{Area}(\gamma_A) + \text{Area}(\gamma_B) - \text{Area}(\gamma_{A \cup B})], \quad (4)$$

where  $\text{Area}(\gamma_{A \cup B})$  is the area of the minimal surface related to  $A \cup B$ . Recently, the holographic mutual information (4) has been considered in a quite remarkably amount of different settings [5, 6]. An striking prediction for the holographic MI arises when analyzing the behaviour of the minimal surface  $\gamma_{A \cup B}$ . In [5] it is shown how, for certain distances between the two regions, there are minimal surfaces  $\gamma_{A \cup B}^{\text{con}}$  connecting  $A$  and  $B$ . For those regimes, the holographic MI has a nonzero value proportional to the number of degrees of freedom in the gauge theory lying on the boundary of  $\text{AdS}_{d+2}$ . However, when the separation between the two regions is large enough compared to their sizes, then a disconnected surface  $\gamma_{A \cup B}^{\text{dis}}$  with

$$\text{Area}(\gamma_{A \cup B}^{\text{dis}}) = \text{Area}(\gamma_A) + \text{Area}(\gamma_B), \quad (5)$$

is both topologically allowed and minimal. In this case, (3) yields  $S_{A \cup B} = S_A + S_B$  and a sharp vanishing of  $I_{AB}$  then occurs. This result is quite surprising from a quantum information point of view since, when the MI vanish, the reduced density matrix  $\rho_{A \cup B}$  factorizes into  $\rho_{A \cup B} = \rho_A \otimes \rho_B$ , implying that the two regions are completely decoupled from each other and thus, all the correlations (both classical and quantum) between  $A$  and  $B$  should be rigorously zero. Indeed, it seems, at least counterintuitive, that all the correlations should strictly vanish at a critical distance, in a field theory in its large  $N$  (t'Hooft coupling) limit. This behaviour is a general prediction of the Ryu-Takayanagi formula (3) which is valid for any two regions of any holographic theory [5].

## MINIMAL SURFACES, WILSON LOOPS AND MUTUAL INFORMATION

At large  $N$  and t'Hooft coupling  $\lambda$ , the holographic dual of a gauge theory is in the small string tension and curvature limit, in which the Wilson loops are represented by classical minimal area worldsheets  $\gamma$  ending on the boundary of AdS, and their expectation values are given by,

$$\langle W(\mathcal{C}) \rangle = \exp \left[ -\sqrt{\lambda} \text{Area}(\gamma) \right], \quad (6)$$

where  $\mathcal{C} = \partial\gamma$  and  $\lambda$  is the t'Hooft coupling [3]. The strong subadditivity property of the EE for disjoint regions can be used to prove the concavity of two coplanar and disjoint Wilson loops by replacing,

$$\text{Area}(\gamma) = -\lambda^{-1/2} \log \langle W(\mathcal{C}) \rangle, \quad (7)$$

in (2). Namely, if one defines  $\mathcal{C}_A = \partial A$ ,  $\mathcal{C}_B = \partial B$  and  $\mathcal{C}_{A \cup B} = \partial(A \cup B)$  then obtains,

$$\langle W(\mathcal{C}_A) \rangle \langle W(\mathcal{C}_B) \rangle \leq \langle W(\mathcal{C}_{A \cup B}) \rangle = \langle W(\mathcal{C}_A) W(\mathcal{C}_B) \rangle. \quad (8)$$

In terms of minimal surface areas, (8) can be illustrated as follows [7]: let us consider, two circular Wilson loops  $\mathcal{C}_A$  and  $\mathcal{C}_B$  of radius  $a$  separated by a distance  $L$ . For certain range of the separation  $L$ , the loops are represented in the bulk by a string worldsheet  $\gamma_{A \cup B}^{\text{con}}$  with the two loops at its boundary. However, there is a critical distance  $L_c$  for which the classical string stretched between the two loops becomes unstable and degenerates into two semispheres  $\gamma_A$  and  $\gamma_B$  which reflects in the saturation of the first inequality in (8),

i.e.  $\langle W(\mathcal{C}_A) W(\mathcal{C}_B) \rangle = \langle W(\mathcal{C}_A) \rangle \langle W(\mathcal{C}_B) \rangle$ . This result is equivalent to the vanishing of the mutual information commented above while it is equally puzzling. Namely, the picture presented above is incomplete. Indeed, when the classical worldsheet  $\gamma_{A \cup B}^{\text{con}}$  becomes unstable, it starts to collapse, but before becoming totally disjoint, the two semispheres  $\gamma_A$  and  $\gamma_B$  stay connected by a long thin tube of a string scale  $l_s$  radius. This tube is protected against total collapse by quantum fluctuations on its surface. This tube represents the exchange of light supergravity modes in the bulk of AdS that couple to the worldsheets  $\gamma_A$  and  $\gamma_B$  of the Wilson loops (see Fig1) and therefore, the correlator

$$\langle W(\mathcal{C}_A) W(\mathcal{C}_B) \rangle - \langle W(\mathcal{C}_A) \rangle \langle W(\mathcal{C}_B) \rangle > 0, \quad (9)$$

i.e., does not totally vanish, but results mediated by light mode exchange between worldsheets. According to this picture concerning the correlators of Wilson loops and recalling (4) and (7), then the Ryu-Takanayagi minimal area prescription (4) for the holographic mutual information between two disjoint circular regions  $A$  and  $B$  of radius  $a$  separated by a distance  $L$ , might be recasted in terms of a supergravity calculation for the long distance correlator of Wilson loops in the AdS bulk, yielding,

$$I_{AB} = \frac{\lambda^{-1/2}}{4G_N^{(d+2)}} \log \left[ \frac{\langle W(\mathcal{C}_A) W(\mathcal{C}_B) \rangle}{\langle W(\mathcal{C}_A) \rangle \langle W(\mathcal{C}_B) \rangle} \right]. \quad (10)$$

In CFT, the correlator within the brackets can be calculated from the OPE of the two Wilson loops, i.e.,

$$\begin{aligned} \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} &= \sum_{i,j;m,n} c_i^{(m)} c_j^{(n)} a^{(\Delta_i^{(m)} + \Delta_j^{(n)})} \langle \mathcal{O}_i^{(m)}(L) \mathcal{O}_j^{(n)}(0) \rangle \\ &= \sum_i \left( c_i^{(0)} \right)^2 \left( \frac{a}{L} \right)^{2\Delta_i^{(0)}} + \sum_{i,\{m,n\} \neq 0} c_i^{(m)} c_i^{(n)} a^{(\Delta_i^{(m)} + \Delta_i^{(n)})} \langle \mathcal{O}_i^{(m)}(L) \mathcal{O}_i^{(n)}(0) \rangle. \end{aligned} \quad (11)$$

where in the last equality, the first term refers to the contributions due to primary operators, while the second one contains the contributions from descendants. Here, we have slightly changed the notation to  $W(0) \equiv W(\mathcal{C}_A)$  and  $W(L) \equiv W(\mathcal{C}_B)$ . In [8], this correlator has been computed through the AdS/CFT correspondence. In the supergravity approximation, the correlator of a pair of Wilson loops largely separated in comparison with their size, is the amplitude for the exchange of SUGRA modes  $\Psi$  between the string worldsheets having  $\mathcal{C}_A$  and  $\mathcal{C}_B$  as boundaries. The main contribution to this amplitude comes from the exchange of the lightest modes which correspond to operators of the lowest dimen-

sions in the CFT side (see Fig. 1).

## HOLOGRAPHIC MUTUAL INFORMATION IN THE $\mathcal{N} = 4$ SYM $AdS_5 \times S^5$ DUALITY

In this section we compute the holographic mutual information (4) in the  $\mathcal{N} = 4$  SYM /  $AdS_5 \times S^5$  duality through (10). The long distance correlator between two circular Wilson loops is given by the exchange of light SUGRA modes in the bulk of  $AdS_5$  that couple to the worldsheet of the loops. For 10-dimensional supergravity compactified on  $X_5 \times S^5$ , the

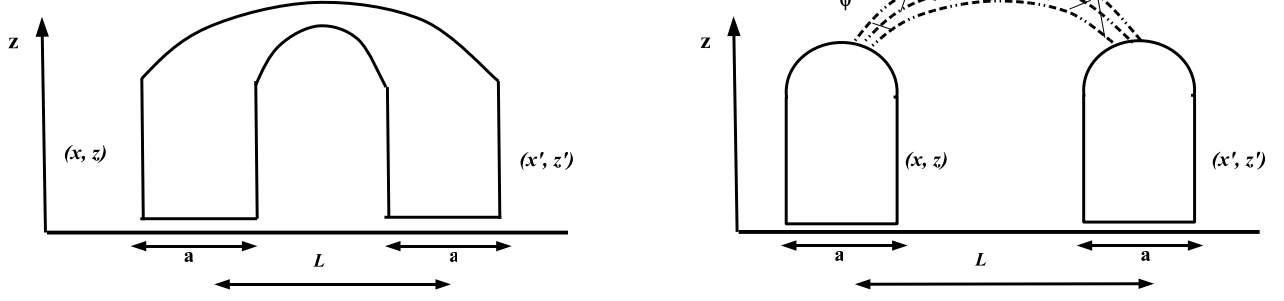


FIG. 1: Correlation function of two disconnected Wilson loops. Left: Regimes of  $a/L$  for which a connected worldsheet  $\gamma_{A \cup B}^{con}$  exists. Right: Regimes of  $a/L$  for which the correlator is calculated through the exchange of bulk supergravity fields  $\Psi$ .

ten-dimensional fields may be written as,

$$\Psi = \sum_{k,I} \phi_k Y_{k,I} \quad (12)$$

where  $\phi_k$  is a five dimensional field and  $Y_{k,I}$  are the spherical harmonics on  $S^5$  with total angular momentum  $k$ . The full spectrum of 10D-supergravity compactified on  $S^5$  was obtained in [10] but, in what follows, we will focus only in the dilaton  $\Phi$  and in "tachyonic" scalars  $s_k$  [8].

The most general form for the long distance correlator integrates the amplitude for the exchange of a SUGRA mode in the bulk between two points on the disjoint worldsheets  $\gamma_A$  and  $\gamma_B$  and then sums over all modes, i.e,

$$\log \left[ \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} \right] = \sum_{i,k,I} Y_{k,I}^2 \quad (13)$$

$$\times \int dA_A \int dA_B g_A^{i,k} g_B^{i,k} G^{i,k}(\sigma, \sigma'),$$

where  $g_A^{i,k}$  and  $g_B^{i,k}$  are the couplings of the field  $i$  to the worldsheets and may be functions of the radial coordinate  $z$ ,  $\sigma = (x, z)$  and  $\sigma' = (x', z')$  are points on the two separated worldsheets,  $G^{i,k}(\sigma, \sigma')$  is the propagator, and  $dA$  are the worldsheet area elements, which are known for a general metric. It is easy to figure out the  $N$  dependence of the amplitude (13). The propagator given by the supergravity calculation is of order  $\sim \alpha'^4 g_s^2$ . This combines with an extra  $\alpha'^{-2}$  coming from the two worldsheet area elements to produce an amplitude (13) of order

$$\alpha'^2 g_s^2 = \frac{1}{4\pi} \frac{g_s}{N}. \quad (14)$$

In [8] (see also [9]), the amplitude (13) was calculated for the exchange of the dilaton  $\Phi$  and the tachyonic scalars  $s_k$ . The leading dependence in  $(a/L)$  of (13) is governed by the asymptotic large  $u = |\sigma - \sigma'|$  of the SUGRA fields Green's function and yields,

$$\log \left[ \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} \right]_s = \frac{1}{4\pi} \frac{g_s}{N} \sum_{k,I} 2^k \pi k Y_{k,I}^2 \left( \frac{a}{L} \right)^{2\Delta}, \quad (15)$$

for the tachyonic scalars  $s_k$  with  $\Delta = k \geq 2$  and,

$$\log \left[ \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} \right]_{\Phi} = \frac{1}{4\pi} \frac{g_s}{N} \sum_{k,I} Y_{k,I}^2 \quad (16)$$

$$\times \frac{2^{k-2} \pi (k+1)(k+2)}{k+3} \left( \frac{a}{L} \right)^{2\Delta},$$

for the dilaton  $\Phi$  with  $\Delta = 4 + k$ , and  $k \geq 0$ .

The most relevant contributions to the amplitude (13) are those obtained when considering the lowest  $k$ 's for each field. Thus, the leading  $(a/L)$  term in (13) is due to the lightest scalar with  $k = 2$  and reads,

$$\log \left[ \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} \right] \sim \frac{1}{4\pi} \frac{g_s}{N} c_2 \left( \frac{a}{L} \right)^4, \quad (17)$$

with

$$c_2 = \left[ 2^k \pi k \sum_I Y_{k,I}^2 \right]_{k=2} = \quad (18)$$

$$\left[ \frac{2^k k (k+3)(k+2)}{\pi (k+1)^2} \right]_{k=2} = \frac{160}{9\pi}.$$

In the last equality we have made use of,

$$\sum_I Y_{k,I}^2 = \frac{(k+3)(k+2)}{\pi^2 (k+1)}, \quad (19)$$

which refers to the summation over the spherical harmonics in  $S^5$ . By substituting (17) into (10) ( $d = 3$ , in the  $\mathcal{N} = 4$  SYM theory), it is straightforward to obtain the  $L \gg a$  asymptotics for the holographic mutual information between two separated circular regions. The result reads as,

$$I_{AB} = \frac{\lambda^{-1/2}}{4G_N^{(d+2)}} \log \left[ \frac{\langle W(L) W(0) \rangle}{\langle W(L) \rangle \langle W(0) \rangle} \right] \sim \frac{\sqrt{\lambda}}{4\pi} c_2 \left( \frac{a}{L} \right)^4, \quad (20)$$

with  $\lambda = g_{YM}^2 N$ . We may further simplify the last expression to obtain,

$$I_{AB} \sim \frac{40}{9\pi^2} \sqrt{\lambda} \left( \frac{a}{L} \right)^4 = 0.450 \sqrt{\lambda} \left( \frac{a}{L} \right)^4. \quad (21)$$

It is worth to note that the  $N$  dependence of  $I_{AB}$  is of order  $\sqrt{N}$ . This implies that the holographic MI between wide separated regions of the (3+1)-dimensional  $\mathcal{N} = 4$  SYM theory, does not suffer a sharp vanishing due to large  $N$  effects but instead, it smoothly decays following a power law given by (21). In this sense, it is convenient to emphasize that recently, by means of field theoretical arguments, it has been shown [11] that the mutual information  $I_{AB}$  between two disjoint compact spatial regions  $A$  and  $B$  in the ground state of a  $d + 1$ -dimensional CFT, in the limit when the separation  $L$  between  $A$  and  $B$  is much greater than their sizes  $a$ , reads,

$$I_{AB} \sim C_{AB} \left( \frac{a}{L} \right)^{2x}, \quad (22)$$

where  $x$  is the smallest scaling dimension of the theory and  $C_{AB}$  is a constant depending on the shape of the regions  $A$  and  $B$ . Concretely, in the case of circular regions in (3+1) dimensional free CFT, the result yields,

$$I_{AB} \sim \frac{4}{15} \left( \frac{a}{L} \right)^4 = 0.26\dot{6} \left( \frac{a}{L} \right)^4, \quad (23)$$

which shows a remarkable agreement with our holographic calculation for the 3+1 dimensional  $\mathcal{N} = 4$  SYM theory.

## CONCLUSION AND DISCUSSION

In this letter we have computed the holographic mutual information (4) between two disconnected circular regions in the (3+1)-dimensional  $\mathcal{N} = 4$  SYM theory dual to  $\text{AdS}_5 \times S^5$  in the limit when the separation  $L$  between them is much larger than their sizes  $a$ . The calculation makes use of previous results concerning the long distance correlator between disconnected Wilson loops in  $\text{AdS}_5$ . Namely, the Ryu-Takayanagi minimal area prescription (4) has been recasted in terms of a supergravity calculation for the long distance correlator of Wilson loops in  $\text{AdS}_5$ . For those regimes in which  $L \gg a$ , this correlator is dominated by the exchange of the lightest SUGRA modes between the worldsheets of the loops. The result contradicts a conjectured sharp vanishing of the holographic mutual information for large  $N$  theories admitting an holographic dual. Nameley, for the regimes that we have considered, the holographic mutual information follows a power law which remarkably coincides with a recent field theoretical prediction for the mutual information between disjoint circular regions in (3+1)-dimensional free CFT [11]. Nevertheless, contrarily to the case of a free theory, the prefactor of the holographic MI in (20), is governed by  $\sqrt{\lambda}$ , with  $\lambda$  the t'Hooft coupling of the boundary theory. We leave for future studies to analyze the phase transition of the holographic mutual information between the regimes in which it follows an area law proportional to the local number of degrees of freedom in the bulk ( $\sim N^2$  in  $\mathcal{N} = 4$  SYM theory) i.e, when a fully connected minimal surface  $\gamma_{A \cup B}^{con}$  is allowed, and the regimes which have been considered here.

## Appendix

In this appendix we show some terms of the  $\mathcal{N} = 4$  SYM/ $\text{AdS}_5$  duality dictionary and the calculations giving the  $N$  dependence of (14) and (20). First, we introduce some basic identities related with the Kaluza Klein compactification of a 10-dimensional supergravity theory into a 5-dimensional one and other identities related with the dictionary of the duality,

$$\begin{aligned} G_N^{10} &= 8\pi^6 \alpha'^4 g_s^2, & G_N^5 &= \frac{G_N^{10}}{\pi^3 R^5} \\ R^4 &= 4\pi g_s \alpha'^2 N \\ g_s &= 2\pi g_{YM}^2 \\ \lambda &= g_{YM}^2 N \end{aligned} \quad (24)$$

where  $\alpha'$  and  $g_s$  are the string tension and string interaction strength respectively,  $R$  is the radius of  $\text{AdS}_5$  (that will be settled to 1 after our calculations),  $N$  is the number of colors of the boundary gauge theory with a  $g_{YM}$  coupling both defining the t'Hooft coupling  $\lambda$ .

With this dictionary entries we first calculate the order of the amplitude (13),

$$\alpha'^2 g_s^2 = \frac{R^4}{4\pi g_s N} g_s^2 = \frac{1}{4\pi} \frac{g_s}{N}, \quad (25)$$

where in the last equality we have settled  $R = 1$ . We now turn to compute the prefactor appearing in (20) setting the order of the holographic mutual information in the long separation regime,

$$\beta = \frac{\lambda^{-1/2}}{4G_N^5} \alpha'^2 g_s^2 c_2, \quad (26)$$

which, using (24) is easily obtained yielding (after settling  $R = 1$ ),

$$\beta = \frac{\sqrt{\lambda}}{4\pi} c_2. \quad (27)$$

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